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Integrability of the 2D Lotka–Volterra system via polynomial first integrals and polynomial inverse integrating factors

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Abstract. We present new first integrals of the two-dimensional Lotka–Volterra systems which have a polynomial inverse integrating factor. Moreover, we characterize all the polynomial first integrals of the two-dimensional Lotka–Volterra systems.

1. Introduction and statement of the results

The two-dimensional Lotka–Volterra dynamical system is defined by

$$\dot{x} = x(a_1 + b_{11}x + b_{12}y) \qquad \dot{y} = y(a_2 + b_{21}x + b_{22}y) \tag{1}$$

where $(a_1, b_{11}, b_{12}, a_2, b_{21}, b_{22})$ are six real (or complex) parameters. This system introduced by Lotka [1] and Volterra [2] appears in chemistry and ecology where it models two species in competition, and it has been widely used in applied mathematics and in a large variety of physical topics such as laser physics, plasma physics, convective instabilities, neural networks, etc. Many authors have examined the integrability of the two-dimensional Lotka–Volterra systems, see for instance Cairó *et al* [3] (who used the Carleman method), Hua *et al* [4] (who used the Hamiltonian method), Cairó and Llibre [5] and Cairó *et al* [6] (who used the Darboux theory of integrability) or the integrability of the three-dimensional Lotka–Volterra systems (see Grammaticos *et al* [7], Almeida *et al* [8] and Cairó and Llibre [9]). These systems have been studied in arbitrary dimension by Cairó *et al* [3] and Cairó and Feix [10].

Recently, Moulin-Ollagnier [11] and Labrunie [12] have characterized the polynomial first integrals of a special three-dimensional Lotka–Volterra system, the so-called ABC system, i.e.

$$\dot{x} = x(Cy + z) \qquad \dot{y} = y(x + Az) \qquad \dot{z} = z(Bx + y).$$

The fact that the vector fields associated with the ABC systems are homogeneous helps in the study of their polynomial first integrals. In general, this is not the case for system (1), but of course this system is simpler than the ABC system as it is two dimensional.

The problem of the integrability of ordinary differential equations is closely related to the problem of finding first integrals. The difficulty of the task was already noted by Poincaré [13] in his discussion of a method to obtain polynomial or rational first integrals. The search for first integrals is a classical tool in the classification of all trajectories of a dynamical system. The following natural question arises: given a system of ordinary differential equations depending on parameters, how does one recognize the values of the parameters for which the system possesses first integrals, or more specifically polynomial first integrals? Many different methods have been used for studying the existence of first integrals. Some of them have been developed for Hamiltonian systems, such as the Ziglin [14, 15] analysis, or the method based on the Noether symmetries [16]. Other methods can be applied to non-Hamiltonian systems: the method of Darboux [17], the method of Lie symmetries [18], Painlevé analysis [19], the use of Lax pairs [20], the direct method [21], the linear compatibility analysis method [22], the Carlemann embedding procedure [3, 10, 23, 24], the Hamiltonian method [4, 25], the quasimonomial formalism [26], etc.

Typically, methods like Carleman or Hamiltonian are based on giving an ansatz to the invariants (time-dependent first integrals). If the goal is only to obtain first integrals (i.e. time-independent first integrals), then the progress in computer algebra now allows us to use successfully the old method consisting in finding first integrals through the use of integrating factors. In fact, once the integrating factor is found then the problem is reduced to a quadrature. So, the first difficulty is in finding the integrating factor, the second being of pure integral calculus. For the first, the method currently used is to give an ansatz, searching integrating factors of a given type (for instance, a polynomial inverse) and solve an algebraic system to determine the coefficients and the conditions that the differential system must satisfy. Let us recall the basic concepts.

By definition a *complex* (respectively, *real*) *planar polynomial differential system* or simply a *polynomial system* will be a differential system of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \dot{x} = P(x, y) \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \dot{y} = Q(x, y) \tag{2}$$

where the dependent variables x and y are complex (respectively, real), the independent one (the *time*) t is real, and P and Q are polynomials in the variables x and y with complex (respectively, real) coefficients. The number $m = \max\{\deg P, \deg Q\}$ denotes the *degree* of the polynomial system. Thus the Lotka–Volterra system (1) is a polynomial system of degree two, with

$$P = x(a_1 + b_{11}x + b_{12}y) \qquad Q = y(a_2 + b_{21}x + b_{22}y). \tag{3}$$

In what follows we will denote the vector field associated with the polynomial system (2) by

$$X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}.$$

We denote by $\mathbb{F}$, either the real field $\mathbb{R}$, or the complex field $\mathbb{C}$; and by $\mathbb{F}$-*polynomial system* the polynomial system (2) with the coefficients of the polynomials P and Q in $\mathbb{F}$. Also we denote by $\mathbb{F}[x, y]$ the ring of polynomials in the variables x and y with coefficients in $\mathbb{F}$.

Here we say that $H : \mathbb{F}^2 \to \mathbb{F}$ is a *first integral* of the $\mathbb{F}$-polynomial system (2) if H is a non-constant function which is constant on all solution curves $(x(t), y(t))$ of system (2); i.e. $H(x(t), y(t)) =$ constant for all values of t for which the solution $(x(t), y(t))$ is defined. Clearly, H is a first integral of system (2) if and only if the function XH is identically zero. If H is a first integral of system (2), then the trajectories of (2) are contained in the curves $H(x, y) = h$ when h varies in $\mathbb{F}$.

Let $R : \mathbb{F}^2 \to \mathbb{F}$ be an analytic function which is not identically zero. The function R is an *integrating factor* of the $\mathbb{R}$-polynomial system (2) if the following three equivalent conditions holds:

$$\frac{\partial(RP)}{\partial x} = -\frac{\partial(RQ)}{\partial y} \quad \Leftrightarrow \quad \mathrm{div}(RP, RQ) = 0 \quad \Leftrightarrow \quad XR + R\,\mathrm{div}(P, Q) = 0. \tag{4}$$

As usual the divergence of the vector field X is defined by

$$\mathrm{div}(X) = \mathrm{div}(P, Q) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}.$$

The *first integral H associated with the integrating factor R* is given by

$$H(x, y) = \int R(x, y)P(x, y)\,\mathrm{d}y + h(x)$$

where $h(x)$ is a function satisfying $\partial H/\partial x = -RQ$. Then

$$\dot{x} = RP = \frac{\partial H}{\partial y} \qquad \dot{y} = RQ = -\frac{\partial H}{\partial x}. \tag{5}$$

Conversely, given a first integral H of system (2) we can always find an integrating factor R for which (5) holds.

Chavarriga *et al* [27] consider the inverse integrating factor and show that, in general, it is better to work with it instead of working directly with a first integral or an integrating factor to study the integrability of a given two-dimensional differential system.

Let $V : \mathbb{F}^2 \to \mathbb{F}$ be an analytic function which is not identically zero. The function V is an *inverse integrating factor* of the $\mathbb{R}$-polynomial system (2) if the following condition holds:

$$XV - V\,\mathrm{div}(P, Q) = 0 \tag{6}$$

which is simply (4) written with $R = 1/V$.

The paper is divided into two parts. In the first part we study the first integrals of the two-dimensional Lotka–Volterra systems which have a polynomial inverse integrating factor, and in the second part we characterize all the polynomial first integrals of the two-dimensional Lotka–Volterra systems via a polynomial integrating factor.

The results of the first part are given in the next theorem, where we omit the symmetric cases that can be obtained under the symmetry $(x, a_1, b_{11}, b_{12}, y, a_2, b_{21}, b_{22}) \to (y, a_2, b_{22}, b_{21}, x, a_1, b_{12}, b_{11})$. We must mention that the first integrals presented in the next theorem have been obtained only using polynomial inverse integrating factors of degree at most five.

Theorem 1. *System (1) possess a first integral H if it satisfies the conditions:*

(a) $r_{12} = a_1 b_{22}(b_{21} - b_{11}) + a_2 b_{11}(b_{12} - b_{22}) = 0$ *and* $a_1 a_2 b_{11} b_{22}(b_{12} - b_{22})(b_{21} - b_{11}) \neq 0$, *and then*

$$H = x^{b_{22}(b_{21}-b_{11})} y^{b_{11}(b_{12}-b_{22})} (a_1 a_2 + a_2 b_{11} x + a_1 b_{22} y)^{b_{11}b_{22}-b_{12}b_{21}}$$

(b) $b_{12} = b_{21} = 0$ *and* $a_1 a_2 (a_1 + a_2) \neq 0$, *and then*

$$H = x^{a_2} y^{-a_1} (a_1 + b_{11} x)^{-a_2} (a_2 + b_{22} y)^{a_1}$$

(c) $b_{12} = -b_{22}$, $b_{21} = 0$ *and* $a_1 a_2 (a_1 + a_2)(a_1 + 2a_2) b_{22} \neq 0$, *and then*

$$H = x^{a_2} y^{-a_1} (a_2 + b_{22} y)^{a_1 + a_2} [(a_1 + a_2)(a_1 + b_{11} x) + b_{11} b_{22} x y]^{-a_2}$$

(d) $a_2 = a_1$, $b_{21} = 0$ *and* $b_{22}(b_{12} - b_{22})(b_{12} - 2b_{22}) \neq 0$, *and then*

$$H = x^{b_{22}} (a_1 + b_{22} y)^{b_{22} - b_{12}} [b_{11} x + (b_{12} - b_{22}) y]^{-b_{22}}$$

(e) $a_1 = a_2 = 0$, $[b_{22}(b_{21} - b_{11}) + b_{11}(b_{12} - b_{22})](b_{11} b_{22} - b_{12} b_{21}) = 0$ *and* $b_{11} b_{22} (b_{12} - b_{22})(b_{21} - b_{11}) \neq 0$, *so*

$$H = x^{b_{22}(b_{21} - b_{11})} y^{b_{11}(b_{12} - b_{22})} (b_{11} x + b_{22} y)^{b_{11} b_{22} - b_{12} b_{21}}$$

(f) $a_1 = a_2$, $b_{21} = b_{22} = 0$ *and* $a_1 b_{12} \neq 0$, *and then*

$$H = x^{-a_1} (b_{11} x + b_{12} y)^{a_1} \mathrm{e}^{b_{12} y}$$

(g) $a_1 = a_2$, $b_{21} = 0$, $b_{12} = b_{22}$ *and* $b_{22} \neq 0$, *and then*

$$H = (a_1 + b_{22} y)^{b_{11}} \mathrm{e}^{b_{22} y / x}$$

(h) $a_1 = a_2$, $b_{12} = 3b_{22}$, $b_{21} = 3b_{11}$ *and* $a_1 b_{11} b_{22} \neq 0$, *and then*

$$H = [a_1(b_{11} x + b_{22} y) + (b_{11} x - b_{22} y)^2](b_{11} x - b_{22} y)^{-1} [(a_1 + b_{11} x)^2 + b_{22} y (2a_1 - 2b_{11} x + b_{22} y)]^{-1/2}$$

(i) $a_1 = a_2$, $b_{12} = 3b_{22}$, $b_{21} = 0$ *and* $a_1 b_{11} b_{22} \neq 0$, *and then*

$$H = y[b_{11} x (2a_1 + b_{22} y) + 2(a_1 + b_{22} y)^2](a_1 + b_{22} y)^{-2} (b_{11} x + 2b_{22} y)^{-1}$$

(j) $a_1 = a_2$, $b_{12} = 3b_{22}$, $b_{11} = 2b_{21}$ *and* $a_1 b_{11} \neq 0$, *and then*

$$H = y^2 [a_1 b_{11} x + (a_1 + b_{22} y)^2][a_1 b_{11} x + 2b_{22} y (a_1 + b_{22} y)]^{-2}$$

(k) $a_1 = -a_2$, $b_{12} = -b_{22}$, $b_{21} = 0$ *and* $a_1 b_{11} b_{22} \neq 0$, *and then*

$$H = y^{-1} (a_1 - b_{22} y) \exp\left[\frac{a_1(a_1 + b_{11} x)}{b_{11} b_{22} x y}\right].$$

The theorem is stated when system (1) is complex. If this system is real, then all the functions of the form $f(x, y)^a$ *that appear in the expression of the first integrals must be* $|f(x, y)|^a$.

Theorem 1 will be proved in section 2. We remark that the first integral given in statement (a) exists for a subclass of codimension one inside the six-dimensional space of the Lotka–Volterra systems (1), the first integrals (b)–(d) exist for a subclass of codimension two, and the remaining first integrals of theorem 1 exist for a subclass of codimension three. Note also that the first integral (a) was already known [3]. Of course, if we had searched polynomial inverse integrating factors of degree larger than five, we would have obtained more new first integrals for the two-dimensional Lotka–Volterra systems.

The next theorem concerns our main result characterizing all polynomial first integrals of the two-dimensional Lotka–Volterra systems.

Theorem 2. *Suppose that the two-dimensional Lotka–Volterra system is not trivial, i.e. that* $b_{11}^2 + b_{12}^2 + b_{21}^2 + b_{22}^2 \neq 0$. *Then given a non-negative integer* n *system (1) has a polynomial first integral of degree* $n + 3$ *if and only if one of the following cases holds:*

$$ib_{12} + (n + 3 - i)b_{22} = 0 \qquad (n + 3 - j)b_{11} + jb_{21} = 0 \qquad ia_1 + ja_2 = 0 \tag{7}$$

with $i \in \{1, \ldots, n + 1\}$ *and* $j \in \{1, \ldots, n + 2 - i\}$. *Moreover, under assumptions (7) the first integral is*

$$H = x^i y^j \, [i(a_1 + b_{11}x) - jb_{22}y]^{n+3-i-j} . \tag{8}$$

Note that for each n if we vary i and j there are $(n + 1)(n + 2)/2$ polynomial first integrals of degree $n + 3$.

Theorem 2 shows that all polynomial first integrals of the two-dimensional Lotka–Volterra systems are particular cases of the first integrals (a) and (e) of theorem 1. We will prove theorem 2 in section 3. The key point in our proof will be the employment of the polynomial integrating factor associated with a polynomial first integral of a polynomial differential system.

2. Polynomial inverse integrating factors

Here we use only *polynomial inverse* integrating factors up to degree five to obtain some new first integrals of the two-dimensional Lotka–Volterra systems (1).

We write $V = \sum_{i+j=0}^{n} a_{i,j} x^i y^j$. Then, the identically zero polynomial (6) of degree $n + 1$ provides the following linear system of $(n + 2)(n + 3)/2$ equations in the $(n + 1)(n + 2)/2$ variables $a_{i,j}$ (the coefficients of V). Each equation of this system comes from identifying to zero every coefficient of (6). Thus the coefficient corresponding to the monomial $x^i y^j$ is

$$a_{i,j}[(i - 1)a_1 + (j - 1)a_2] + a_{i,j-1}[(i - 1)b_{12} + (j - 3)b_{22}]$$
$$+a_{i-1,j}[(i - 3)b_{11} + (j - 1)b_{21}].$$

If some of the coefficients $a_{l,m}$ which appear in the last equation do not appear in V, then they must be taken equal to zero. Therefore, the complete linear system is

$$
\begin{array}{llll}
a_{0,0}(-a_1-a_2) & & & =0\\
a_{1,0}(-a_2) & & +a_{0,0}(-2b_{11}-b_{21}) & =0\\
a_{0,1}a_1 & +a_{0,0}(b_{12}+2b_{22}) & & =0\\
a_{2,0}(a_1-a_2) & & +a_{1,0}(-b_{11}-b_{21}) & =0\\
 & a_{1,0}(-2b_{22}) & +a_{0,1}(-2b_{11}) & =0\\
a_{0,2}(-a_1+a_2) & +a_{0,1}(-b_{12}-b_{22}) & & =0\\
a_{3,0}(2a_1-a_2) & & +a_{2,0}(-b_{21}) & =0\\
a_{2,1}a_1 & +a_{2,0}(b_{12}-2b_{22}) & +a_{1,1}(-b_{11}) & =0\\
a_{1,2}a_2 & +a_{1,1}(-b_{22}) & +a_{0,2}(-2b_{11}+b_{21}) & =0\\
a_{0,3}(-a_1+2a_2) & +a_{0,2}(-b_{12}) & & =0\\
\vdots & \vdots & \vdots & =0\\
a_{0,n}(-a_1+(n-1)a_2) & +a_{0,n-1}(-b_{12}+(n-3)b_{22}) & & =0\\
 & & a_{n,0}((n-2)b_{11}-b_{21}) & =0\\
 & a_{n,0}((n-1)b_{12}-2b_{22}) & +a_{n-1,1}((n-3)b_{11}) & =0\\
 & a_{n-1,1}((n-2)b_{12}-b_{22}) & +a_{n-2,2}((n-4)b_{11}+b_{21}) & =0\\
 & \vdots & \vdots & =0\\
 & a_{1,n-1}((n-3)b_{22}) & +a_{0,n}(-2b_{11}+(n-1)b_{21}) & =0\\
 & a_{0,n}(-b_{12}+(n-2)b_{22}) & & =0.
\end{array}
$$

We are searching here for the polynomial inverse integrating factors V of degree n. So, in order for the linear subsystem formed by the last $n+2$ equations in the $n+1$ variables $a_{n,0}, a_{n-1,1}, \ldots, a_{0,n}$ to have a non-zero solution, the $(n+2)\times(n+1)$ matrix M of this subsystem must have rank smaller than $n+1$. Hence, the $n+2$ determinants of the submatrices $(n+1)\times(n+1)$ of M obtained by omitting a row of M must be zero, i.e.

$$
\begin{array}{lll}
\prod_{k=1}^{n+1}((n-1-k)b_{11}+(k-2)b_{21}) & & =0\\
\left[\prod_{k=1}^{n}((n-1-k)b_{11}+(k-2)b_{21})\right]\times(-b_{12}+(n-2)b_{22}) & & =0\\
\left[\prod_{k=1}^{n-1}((n-1-k)b_{11}+(k-2)b_{21})\right] & \times\left[\prod_{l=1}^{2}((l-2)b_{12}+(n-1-l)b_{22})\right] & =0\\
\vdots & \vdots & =0\\
((n-2)b_{11}-b_{21}) & \times\left[\prod_{l=1}^{n}((l-2)b_{12}+(n-1-l)b_{22})\right] & =0\\
 & \prod_{l=1}^{n+1}((l-2)b_{12}+(n-1-l)b_{22}) & =0.
\end{array}
\tag{9}
$$

If $n \neq 3$ then the solutions of this last system are

$$(i-2)b_{12} + (n-1-i)b_{22} = 0 \qquad (n-1-j)b_{11} + (j-2)b_{21} = 0 \quad (10)$$

with

$$i \in \{1, \ldots, n+1\} \qquad j \in \{1, \ldots, n+2-i\}. \quad (11)$$

So we have $(n+1)(n+2)/2$ different solutions.

If $n = 3$ then the determinant of system (9) is identically zero.

Proof of theorem 1. First we search the polynomial inverse integrating factors of degree three. Taking all the coefficients a_{ij} of V zero except for the coefficients a_{11}, a_{21} and a_{12}, the above linear system becomes

$$\begin{aligned} &-a_{11}b_{11} + a_1 a_{21} = 0 \\ &-a_{11}b_{22} + a_2 a_{12} = 0 \\ &a_{21}(b_{12} - b_{22}) + a_{12}(b_{21} - b_{11}) = 0 \end{aligned} \quad (12)$$

which is a linear system with three equations and three variables a_{11}, a_{21} and a_{12}. The solution of (12) is not identically zero if the determinant of the a_{ij}, i.e. $r_{12} = a_1 b_{22}(b_{21} - b_{11}) + a_2 b_{11}(b_{12} - b_{22})$ is zero. Then, its solution is $a_{11} = a_1 a_2$, $a_{21} = a_2 b_{11}$ and $a_{12} = a_1 b_{22}$ if $a_1 a_2 b_{11} b_{22}(b_{12} - b_{22})(b_{21} - b_{11}) \neq 0$. Hence

$$V = xy(a_1 a_2 + a_2 b_{11} x + a_1 b_{22} y).$$

Using this inverse integrating factor we obtain the first integral (a).

Searching other polynomial inverse integrating factors of degree three we find

$$V = x(a_1 + b_{22} y)[b_{11} x + (b_{12} - b_{22}) y]$$

if $a_1 = a_2$, $b_{21} = 0$ and $b_{22}(b_{12} - b_{22})(b_{12} - 2b_{22}) \neq 0$;

$$V = xy(b_{11} x + b_{22} y)$$

if $a_1 = a_2 = 0$, $[b_{22}(b_{21} - b_{11}) + b_{11}(b_{12} - b_{22})](b_{11} b_{22} - b_{12} b_{21}) = 0$ and $b_{11} b_{22}(b_{12} - b_{22})(b_{21} - b_{11}) \neq 0$; and

$$V = x^2(a_2 + b_{22} y)$$

if $a_1 = a_2$, $b_{21} = 0$, $b_{12} = b_{22}$ and $b_{22} \neq 0$. These different inverse integrating factors V of degree three provide the first integrals (d), (e) and (g), respectively.

Now we search the polynomial inverse integrating factors of degree $n = 0, 1, 2, 4, 5$ which provide integrable Lotka–Volterra systems different from the systems with first integrals (a), (b), (e) and (g). Using the above linear system, it is easy to see that there are no new integrable cases having a polynomial inverse integrating factor of degree $n = 0$ or 1.

Searching the polynomial inverse integrating factors of degree four we obtain

$$V = xy(a_1 + b_{11} x)(a_2 + b_{22} y)$$

if $b_{12} = b_{21} = 0$ and $a_1 a_2(a_1 + a_2) \neq 0$. This V provides the first integral (b).

Computing polynomial inverse integrating factors of degree two which correspond to new integrable systems, we obtain

$$V = x(b_{11} x + b_{12} y)$$

if $a_1 = a_2$, $b_{21} = b_{22} = 0$ and $a_1 b_{12} \neq 0$. This V provides the first integral (f).

Searching polynomial inverse integrating factors of degree five which give new integrable systems, we find

$$V = xy(a_2 + b_{22}y)[(a_1 + a_2)(a_1 + b_{11}x) + b_{11}b_{22}xy]$$

if $b_{12} = -b_{22}$, $b_{21} = 0$ and $a_1a_2(a_1 + a_2)(a_1 + 2a_2)b_{22} \neq 0$;

$$V = (b_{11}x - b_{22}y)[a_1(b_{11}x + b_{22}y) + (b_{11}x - b_{22}y)^2]$$
$$\times[(a_1 + b_{11}x)^2 + b_{22}y(2a_1 - 2b_{11}x + b_{22}y)]$$

if $a_1 = a_2$, $b_{12} = 3b_{22}$, $b_{21} = 3b_{11}$ and $a_1b_{11}b_{22} \neq 0$;

$$V = y(a_1 + b_{22}y)(b_{11}x + 2b_{22}y)[b_{11}x(2a_1 + b_{22}y) + 2(a_1 + b_{22}y)^2]$$

if $a_1 = a_2$, $b_{12} = 3b_{22}$, $b_{21} = 0$ and $a_1b_{11}b_{22} \neq 0$;

$$V = y[a_1b_{11}x + (a_1 + b_{22}y)^2][a_1b_{11}x + 2b_{22}y(a_1 + b_{22}y)]$$

if $a_1 = a_2$, $b_{12} = 3b_{22}$, $b_{11} = 2b_{21}$ and $a_1b_{11} \neq 0$;

$$V = x^2y^2(a_1 - b_{22}y)$$

if $a_1 = -a_2$, $b_{12} = -b_{22}$, $b_{21} = 0$ and $a_1b_{11}b_{22} \neq 0$. These different inverse integrating factors V of degree five provide the first integrals (c), (h)–(k), respectively, and completes the proof of theorem 1. □

We note first that statement (a) is the invariant III of Cairó *et al* [3, 10] and second that statements (d)–(j) concern cases with the condition $a_1 = a_2$, which is the condition of having a time-dependent first integral, which is the invariant II of the same authors. We see that it is required at least one additional condition to have an integrable system.

3. Polynomial integrating factors

This section is devoted to proving theorem 2.

Proof of theorem 2. We assume that H is a polynomial first integral of degree $n + 3$ for the two-dimensional Lotka–Volterra system (1). Therefore, from (5), it follows that the integrating factor R is a polynomial of degree n. We write $R = \sum_{i+j=0}^{n} a_{i,j}x^iy^j$. Then, the identically zero polynomial $XR + R\,\mathrm{div}(P, Q)$ of degree $n + 1$ provides the following linear system of $(n + 2)(n + 3)/2$ equations in the $(n + 1)(n + 2)/2$ variables $a_{i,j}$ (the coefficients of R). Each equation of this system comes from identifying with zero every coefficient of (4). Thus the coefficient corresponding to the monomial x^iy^j is

$$a_{i,j}[(i + 1)a_1 + (j + 1)a_2] + a_{i,j-1}[(i + 1)b_{12} + (j + 1)b_{22}] + a_{i-1,j}[(i + 1)b_{11} + (j + 1)b_{21}].$$

Of course, if some of the coefficients $a_{l,m}$ which appear in the last equation do not appear in

R, then they must be taken equal to zero. Therefore, the complete linear system is

$$
\begin{array}{llll}
a_{0,0}(a_1+a_2) & & & =0\\
a_{1,0}(2a_1+a_2) & & +\,a_{0,0}(2b_{11}+b_{21}) & =0\\
a_{0,1}(a_1+2a_2) & +\,a_{0,0}(b_{12}+2b_{22}) & & =0\\
a_{2,0}(3a_1+a_2) & & +\,a_{1,0}(3b_{11}+b_{21}) & =0\\
a_{1,1}(2a_1+2a_2) & +\,a_{1,0}(2b_{12}+2b_{22}) & +\,a_{0,1}(2b_{11}+2b_{21}) & =0\\
a_{0,2}(a_1+3a_2) & +\,a_{0,1}(b_{12}+3b_{22}) & & =0\\
a_{3,0}(4a_1+a_2) & & +\,a_{2,0}(4b_{11}+b_{21}) & =0\\
a_{2,1}(3a_1+2a_2) & +\,a_{2,0}(3b_{12}+2b_{22}) & +\,a_{1,1}(3b_{11}+2b_{21}) & =0\\
a_{1,2}(2a_1+3a_2) & +\,a_{1,1}(2b_{12}+3b_{22}) & +\,a_{0,2}(2b_{11}+3b_{21}) & =0\\
a_{0,3}(a_1+4a_2) & +\,a_{0,2}(b_{12}+4b_{22}) & & =0\\
\vdots & \vdots & \vdots & =0\\
a_{0,n}(a_1+(n+1)a_2) & +\,a_{0,n-1}(b_{12}+(n+1)b_{22}) & & =0\\
 & & a_{n,0}((n+2)b_{11}+b_{21}) & =0\\
 & a_{n,0}((n+1)b_{12}+2b_{22}) & +\,a_{n-1,1}((n+1)b_{11}+2b_{21}) & =0\\
 & a_{n-1,1}(nb_{12}+3b_{22}) & +\,a_{n-2,2}(nb_{11}+3b_{21}) & =0\\
 & \vdots & \vdots & =0\\
 & a_{1,n-1}(2b_{12}+(n+1)b_{22}) & +\,a_{0,n}(2b_{11}+(n+1)b_{21}) & =0\\
 & a_{0,n}(b_{12}+(n+2)b_{22}) & & =0.
\end{array}
$$

Since we are searching the polynomial integrating factors R of degree n, in order that the linear subsystem formed by the last $n+2$ equations in the $n+1$ variables $a_{n,0}, a_{n-1,1}, \ldots, a_{0,n}$ has a non-zero solution, the $(n+2)\times(n+1)$ matrix M of this subsystem must have rank smaller than $n+1$. Hence, the $n+2$ determinants of the submatrices $(n+1)\times(n+1)$ of M obtained by omitting a row of M must be zero, i.e.

$$
\begin{array}{lll}
\displaystyle\prod_{k=1}^{n+1}((n+3-k)b_{11}+kb_{21}) & & =0\\
\left[\displaystyle\prod_{k=1}^{n}((n+3-k)b_{11}+kb_{21})\right] & \times(b_{12}+(n+2)b_{22}) & =0\\
\left[\displaystyle\prod_{k=1}^{n-1}((n+3-k)b_{11}+kb_{21})\right] & \left[\displaystyle\prod_{l=1}^{2}(lb_{12}+(n+3-l)b_{22})\right] & =0\\
\vdots & \vdots & =0\\
((n+2)b_{11}+b_{21}) & \times\left[\displaystyle\prod_{l=1}^{n}(lb_{12}+(n+3-l)b_{22})\right] & =0\\
 & \displaystyle\prod_{l=1}^{n+1}(lb_{12}+(n+3-l)b_{22}) & =0.
\end{array}
$$

The solutions of this last system are

$$ib_{12} + (n+3-i)b_{22} = 0 \qquad (n+3-j)b_{11} + jb_{21} = 0 \tag{13}$$

with

$$i \in \{1, \ldots, n+1\} \qquad j \in \{1, \ldots, n+2-i\}. \tag{14}$$

So we have $(n+1)(n+2)/2$ different solutions. By assumption $b_{11}^2 + b_{12}^2 + b_{21}^2 + b_{22}^2 \neq 0$, and we select one solution of (13) and (14). Then by inspection we see that all the combinations $\alpha b_{12} + \beta b_{22}$, $\gamma b_{11} + \delta b_{21}$ which appear in the first $(n+1)(n+2)/2$ equations of the complete linear system are not zero. Moreover, if all the expressions $ka_1 + la_2$ which appear in the linear system are not zero, it follows easily that $R \equiv 0$. So at least one of these expressions must be zero. If two or more of them are zero, then $a_1 = a_2 = 0$. In this last case we can compute the polynomial first integral, which coincides with (e) of theorem 1. If only one expression $ka_1 + la_2$ is zero, then in order to avoid $R \equiv 0$ we must take

$$ia_1 + ja_2 = 0. \tag{15}$$

Now by applying the first integral (a) of theorem 1, theorem 2 follows. □

Note that the case $a_1 = a_2 = 0$ is contained in (15). When (13)–(15) are satisfied, it is not difficult to verify that

$$R = x^{i-1}y^{j-1}\left[i(a_1 + b_{11}x) - jb_{22}y\right]^{n+2-i-j}.$$

4. Conclusion

In this work we prove that the use of an ansatz on the integrating factor, instead of on the first integral, is quite productive in first integrals for a given two-dimensional differential system. As a matter of fact, using only polynomial inverse integrating factors up to degree five, we have obtained 10 new first integrals for the two-dimensional Lotka–Volterra system. Moreover, the use of polynomial integrating factors has been useful to establish a general proof of the fact that only two classes of polynomial first integrals (cases (a) and (e) of theorem 1) exist for this system. With the help of computer algebra, the way is now open to search new cases of integrability with inverse integrating factors of degree higher than five.

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